Solutions to Problems in Introduction to Topology by Bert Mendelson (Chapter 2)

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July 29, 2019

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2.1

N/A

2.2

2.2.1

We need to show that $d_k : X \times X \to \mathbb{R}$ defined as $d_k(x, y) = kd(x, y)$ satisfies the 4 conditions for metric spaces. Observe that for any $x, y, z \in X$:

1. Since $k > 0$ and $d : X \times X \to \mathbb{R}$ is a metric, it follows that $d_k(x, y) = kd(x, y) \geq 0$
2. $d_k(x, y) = 0 \iff kd(x, y) = 0 \iff d(x, y) = 0 \iff x = y$
3. $d_k(x, y) = kd(x, y) = kd(y, x) = d_k(y, x)$
4. $d_k(x, z) = kd(x, z) \leq k(d(x, y) + d(y, z)) = kd(x, y) + kd(y, z) = d_k(x, y) + d_k(y, z)$

Thus, (X, d) is a metric space.

2.2.2

We are told that $d'' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as $d''(x, y) = \sum\limits_{i=1}^{n} |x_i - y_i|$. Observe that for any $x, y, z \in \mathbb{R}^n$:

1. $d''(x, y) = \sum\limits_{i=1}^{n} |x_i - y_i| \geq 0 \; \forall x, y \in \mathbb{R}^n$ since $|a - b| \geq 0 \; \forall a, b \in \mathbb{R}$
2. $d''(x, y) = 0 \iff \sum\limits_{i=1}^{n} |x_i - y_i| = 0 \iff x_i = y_i \; \forall i \in [n] \iff x = y$
3. $d''(x, y) = \sum\limits_{i=1}^{n} |x_i - y_i| = \sum\limits_{i=1}^{n} |y_i - x_i| = d''(y, x)$
4. $d''(x, z) = \sum\limits_{i=1}^{n} |x_i - z_i| = \sum\limits_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq \sum\limits_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = \sum\limits_{i=1}^{n} |x_i - y_i| + \sum\limits_{i=1}^{n} |y_i - z_i| = d''(x, y) + d''(y, z)$

Hence, $(\mathbb{R}^n, d'')$ is a metric space.

2.2.3

Observe that

$$\begin{aligned}\big(d(x,y)\big)^2 &= \left(\max_{1\le i\le n}\{|x_i-y_i|\}\right)^2\\ &= \max_{1\le i\le n}\{|x_i-y_i|^2\}\\ &\le \sum_{i=1}^n |x_i-y_i|^2\\ &= \sum_{i=1}^n (x_i-y_i)^2\\ &= \big(d'(x,y)\big)^2\end{aligned}$$

$\Rightarrow d(x,y) \le d'(x,y)$. Moreover,

$$\begin{aligned}d'(x,y) &= \sqrt{\sum_{i=1}^n (x_i-y_i)^2}\\ &= \sqrt{\sum_{i=1}^n |x_i-y_i|^2}\\ &\le \sqrt{n\left(\max_{1\le i\le n}\{|x_i-y_i|\}\right)}\\ &= \sqrt{n}\sqrt{\max_{1\le i\le n}\{|x_i-y_i|\}}\\ &= \sqrt{n}\cdot d(x,y)\end{aligned}$$

Thus, $d(x,y) \le d'(x,y) \le \sqrt{n}\cdot d(x,y)$.

The next set of inequalities is easier to see, but note that

$$d(x,y) = \max_{1\le i\le n}\{|x_i-y_i|\} \le \sum_{i=1}^n |x_i-y_i| = d''(x,y)$$

and

$$d''(x,y) = \sum_{i=1}^n |x_i-y_i| \le n\left(\max_{1\le i\le n}\{|x_i-y_i|\}\right) = n\cdot d(x,y)$$

Hence, $d(x,y) \le d''(x,y) \le n\cdot d(x,y)$.

2.2.4

We are told that $d : C^0([a,b]) \times c^0([a,b]) \to \mathbb{R}$ is defined as $d(f,g) = \int_a^b |f(t)-g(t)|dt$. Observe that for any $f,g,h \in C^0([a,b])$:

1. $d(f,g) = \int_a^b |f(t)-g(t)|dt \ge 0$ since $|f(t)-g(t)| \ge 0\ \forall t \in [a,b]$
2.

$$\begin{aligned}d(f,g) = 0 &\iff \int_a^b |f(t)-g(t)|dt = 0\\ &\iff |f(t)-g(t)| = 0\ \forall t\in[a,b]\\ &\iff f(t) = g(t)\ \forall t\in[a,b]\end{aligned}$$

3.

$$d(f,g)=\int_a^b |f(t)-g(t)|dt=\int_a^b \big|(-1)\big(g(t)-f(t)\big)\big|\,dt=\int_a^b |g(t)-f(t)|dt=d(g,f)$$

4.

$$\begin{aligned}
d(f,h) &= \int_a^b |f(t)-h(t)|dt \\
&= \int_a^b |f(t)-g(t)+g(t)-h(t)|dt \\
&\leq \int_a^b \big(|f(t)-g(t)|+|g(t)-h(t)|\big)dt \\
&= \int_a^b |f(t)-g(t)|dt+\int_a^b |g(t)-h(t)|dt \\
&= d(f,g)+d(g,h)
\end{aligned}$$

Hence, $(C^0([a,b]),d)$ is a metric space.

2.2.5

Note that $C^b(X)$ is the set of all bounded functions defined on the set X. We are told that $d' : C^b([a,b])\times C^b([a,b]) \to \mathbb{R}$ is defined as $d'(f,g)=\sup_{x\in[a,b]}\{|f(x)-g(x)|\}$. Observe that for any $f,g,h \in C^b([a,b])$:

1. $d'(f,g)=\sup_{x\in[a,b]}\{|f(x)-g(x)|\} \geq 0$ since $|f(x)-g(x)| \geq 0 \;\forall x \in [a,b]$

2.

$$\begin{aligned}
d'(f,g)=0 &\iff \sup_{x\in[a,b]}\{|f(x)-g(x)|\}=0 \\
&\iff |f(x)-g(x)|=0 \;\forall x\in[a,b] \;\big(\text{again, because } |f(x)-g(x)|\geq 0 \;\forall x\in[a,b]\big) \\
&\iff f(x)=g(x) \;\forall x\in[a,b]
\end{aligned}$$

3.

$$d'(f,g)=\sup_{x\in[a,b]}\{|f(x)-g(x)|\}=\sup_{x\in[a,b]}\big\{\big|(-1)\big(g(x)-f(x)\big)\big|\big\}=\sup_{x\in[a,b]}\{|g(x)-f(x)|\}=d'(g,f)$$

4.

$$\begin{aligned}
d'(f,h) &= \sup_{x\in[a,b]}\{|f(x)-h(x)|\} \\
&= \sup_{x\in[a,b]}\{|f(x)-g(x)+g(x)-h(x)|\} \\
&\leq \sup_{x\in[a,b]}\{|f(x)-g(x)|+|g(x)-h(x)|\} \\
&\big(\text{since } \forall x\in[a,b],\ |f(x)-h(x)|\leq |f(x)-g(x)|+|g(x)-h(x)| \;\big(\text{by triangle inequality for real numbers}\big) \\
&= \sup_{x\in[a,b]}\{|f(x)-g(x)|\}+\sup_{x\in[a,b]}\{|g(x)-h(x)|\} \\
&= d'(f,g)+d'(g,h)
\end{aligned}$$

Hence, $\big(C^b([a,b]),d'\big)$ is a metric space.

2.2.6

Observe that

$$d(f,g) = \int_a^b |f(t) - g(t)|dt \leq \int_a^b \sup_{t\in[a,b]}\{|f(t) - g(t)|\}dt = \int_a^b d'(f,g)dt = (b-a)d'(f,g)$$

In particular, setting $b := 1$ and $a := 0$, we have $d(f,g) \leq d'(f,g)$.

2.2.7

We are told that $d : X \times X \to \mathbb{R}$ is defined as $d(x,x) = 0$ and $d(x,y) = 1$ for any $x \neq y$. Observe that for any $x, y \in X$:

1. $d(x,y) \geq 0$ by definition

2. $d(x,y) = 0 \iff x = y$ by definition

3. $x = y \iff y = x \Rightarrow d(x,y) = 0 = d(y,x)$. On the other hand, $x \neq y \iff y \neq x \Rightarrow d(x,y) = 1 = d(y,x)$.

4. If $x = z$, then $d(x,z) = 0 \Rightarrow d(x,z) \leq d(x,y) + d(y,z)$ since $d(x,y), d(y,z) \geq 0$. If $x \neq z$, then $d(x,z) = 1$. Let $y \in X$. Then exacly one of the following holds: $(y = x \wedge y \neq z)$, $(y = z \wedge y \neq x)$, or $(y \neq x \wedge y \neq z)$; i.e., we cannot have $x = y = z$ because this would imply $x = z$. Hence, $d(x,y) + d(y,z) \geq 1 \Rightarrow d(x,z) \leq d(x,y) + d(y,z)$.

Thus, (X, d) is a metric space.

2.2.8

Given p prime, we are told that $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is defined as $d(m,n) = 0$ for $m = n$, and $d(m,n) = \frac{1}{p^t}$ for $m \neq n$, where $t = t(m,n)$ is the unique integer such that $m - n = p^t \cdot k$ (where k is not divisible by p). Obsere that for any $m, n, o \in \mathbb{Z}$:

1. $d(m,n) \geq 0$ by definition (since $0 \geq 0$ and given p prime, for any integer t, $\frac{1}{p^t} > 0$)

2. $d(m,n) = 0 \iff m = n$ by definition (again, since given p prime, for any integer t, $\frac{1}{p^t} > 0$)

3. $m = n \iff n = m \Rightarrow d(m,n) = 0 = d(n,m)$. On the other hand, $m \neq n$ implies that $d(m,n) = \frac{1}{p^r}$ where $r = r(m,n)$ is the unique integer such that $m - n = p^r \cdot a$, where $a \in \mathbb{Z}$ such that $a \not| p$, and $d(n,m) = \frac{1}{p^s}$, where $s = s(n,m)$ is the unique integer such that $n - m = p^s \cdot b$, where $b \in \mathbb{Z}$ such that $b \not| p$. Thus, it suffices to show that $r = s$. Observe that $p^r a = m - n = -(n-m) \Rightarrow n - m = -p^r a = p^r(-a) \Rightarrow r = s$. Hence, $d(m,n) = d(n,m)$.

4. We want to show that if $m, n, o \in \mathbb{Z}$, then $d(m,o) \leq d(m,n) + d(n,o)$. $\exists!\ r \in \mathbb{Z}$ such that $m - n = p^r a$, where $a \in \mathbb{Z}$ such that $a \not| p$; similarly, $\exists!\ s \in \mathbb{Z}$ such that $n - o = p^s b$, where $b \in \mathbb{Z}$ such that $b \not| p$. WLOG suppose $s \leq r$. Then $m - o = (m-n) + (n-o) = p^r a + p^s b = p^s(p^{r-s}a + b) \Rightarrow m - o = p^t c$ for some integer $t \geq s$ and $c \in \mathbb{Z}$ such that $c \not| p$. Therefore, $d(m,o) = \frac{1}{p^t} \leq \frac{1}{p^s} = d(n,o) \leq d(m,n) + d(n,o)$.

Thus, $(\mathbb{Z}, d)$ is a metric space.

2.3

2.3.1

We are told that $X = C^0([a,b])$, and we want to prove that $I : (C^0([a,b]), d^*) \to (\mathbb{R}, d)$, with $d^*(f,g) = \int_a^b |f(t) - g(t)|dt$, is continuous. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for any $f, g \in C^0([a,b])$ such that $d^*(f,g) < \delta$, we have:

$$d(I(f), I(g)) = \left| \int_a^b f(t)dt - \int_a^b g(t)dt \right| = \left| \int_a^b \big(f(t) - g(t)\big)dt \right| \leq \int_a^b |f(t) - g(t)|dt = d^*(f,g) < \delta = \epsilon$$

$\Rightarrow I$ is continuous.

2.3.2

We are told that for $i = 1, ...n$, (X_i, d_i) and (Y, d_i') are metric spaces, and that $X = \prod_{i=1}^{n} X_i$ and $Y = \prod_{i=1}^{n} Y_i$. X and Y, equipped, respectively, with the metrics $d_X : X \times X \to \mathbb{R}$ and $d_y : Y \times Y \to \mathbb{R}$, defined as $d_X(x,y) = \max_{1 \leq i \leq n} \{d_i(x_i, y_i)\}$ and $d_Y(x,y) = \max_{1 \leq i \leq n} \{d_i'(x_i, y_i)\}$, are metric spaces. Given that each $f_i : X_i \to Y_i$ are continuous, we want to prove that $F : X \to Y$ defined as $F(x) = F(x_1, ..., x_n) = (f_1(x_1), ..., f_n(x_n))$ is continuous.

Observe that for any $F(x), F(y) \in Y$, $d_Y(F(x), F(y)) = \max_{1 \leq i \leq n} \{d_i'(f_i(x_i), f_i(y_i))\} = d_j'(f_j(x_j), f_j(y_j))$ for some $j \in [n]$. Since each f_i is continuous for $i = 1, ..., n$, this implies that given any $\epsilon > 0$, there exists a $\delta > 0$ such that $d_j'(f_j(x_j), f_j(y_j)) < \epsilon$ whenver $d_j(x_j, y_j) < \delta$. Hence, given $\epsilon > 0$, we can always choose a $\delta > 0$ so that $d_Y(F(x), F(y)) < \epsilon$ whenever $d_X(x,y) < \delta \Rightarrow F : X \to Y$ is continuous.

2.3.3

Given the metrics on $\mathbb{R}^2$ d and d', where d is defined as $d((x_1, x_2), (y_1, y_2)) = \max_{1 \leq i \leq 2} \{|x_i - y_i|\}$ and d' is the normal Euclidean distance, we want to prove that $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = x_1 + x_2$ is continuous.

First we prove that f is continuous with the metric d on $\mathbb{R}^2$. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2}$. Then for any $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ such that $d((x_1, x_2), (y_1, y_2)) < \delta$, we have:

$$|f(x_1, x_2) - f(y_1, y_2)| = |x_1 + x_2 - (y_1 + y_2)| = |(x_1 - y_1) + (x_2 - y_2)| \leq |x_1 - y_1| + |x_2 - y_2| < 2\delta = \epsilon$$

$\Rightarrow f$ is continuous.

Now, we prove that f is continuous with the metric d' on $\mathbb{R}^2$. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2}$. Then observe that for any $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ such that $d'((x_1, x_2), (y_1, y_2)) < \delta$ we have:

$$\begin{aligned} d'((x_1, x_2), (y_1, y_2)) < \delta &\iff \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \delta \\ &\iff (x_1 - y_1)^2 + (x_2 - y_2)^2 < \delta^2 \\ &\Rightarrow (x_1 - y_1)^2 < \delta^2 - (x_2 - y_2)^2 \leq \delta^2 \ \wedge \ (x_2 - y_2)^2 < \delta^2 \\ &\Rightarrow |x_1 - y_1| = \sqrt{(x_1 - y_1)^2} < \delta \ \wedge \ |x_2 - y_2| = \sqrt{(x_2 - y_2)^2} < \delta \end{aligned}$$

$\Rightarrow |f(x_1, x_2) - f(y_1, y_2)| = |x_1 + x_2 - (y_1 + y_2)| \leq |x_1 - y_1| + |x_2 - y_2| < 2\delta = \epsilon \Rightarrow f$ is continuous.

2.3.4

I'm too lazy. Hopefully this omission does not come back to haunt me.

2.4

2.4.1

Recall that $N \subset X$ is a neighborhood of a if N contains an open ball $B(a;\delta) \subset X$ centered at a with some radius $\delta > 0$. Let $\delta := \frac{1}{2}$. Then since $d(a,x) = 1$ for any $x \in X$ such $x \neq a$, $B(a;\delta) = \{a\} \Rightarrow B(a;\delta) \subseteq \{a\} \Rightarrow \{a\}$ is a neighborhood of a. Moreover, $\{a\}$ constitutes a basis for the system of neighborhoods of a since for any neighborhood N of a, $N \ni a$ and $a \in \{a\}$. Now, let S be a subset of X. If $p \in S$, then $\{p\} \subseteq S \Rightarrow S$ is a neighborhhod of p.

2.4.2

To show that $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & x \leq a \\ 1 & x > a \end{cases}$$

is discontinuous at a, we need to show that there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there is $x \in B(a;\delta)$ but $f(x) \notin B(f(a);\epsilon_0)$. Let $\epsilon_0 := \frac{1}{2}$. Then observe that $\left|\left(a + \frac{\delta}{2}\right) - a\right| = \left|\frac{\delta}{2}\right| = \frac{1}{2}|\delta| < \delta$, but $|f(a + \frac{\delta}{2}) - f(a)| = |1 - 0| = 1 \geq \frac{1}{2}$. Hence, $\forall\, \delta > 0\ \exists x \in B(a;\delta)$ such that $f(x) \notin B(a;\epsilon_0)$; namely, $x := a + \frac{\delta}{2}$.

Now, for any $x \in \mathbb{R} \setminus \{0\}$, f is locally constant. Thus, it is clear that at any other point besides a, f is continuous.

2.4.3

($\Rightarrow$) Suppose f is continuous. Then for each neighborhood M of a, $f^{-1}(M)$ is a neighborhood of a. If $N \in \mathcal{B}_{f(a)}$, then N is a neighborhood of $f(a)$; hence, it follows immediatley that $f^{-1}(N)$ is a neighborhood of a.
($\Leftarrow$) Conversely, suppose that for every $N \in \mathcal{B}_{f(a)}$, $f^{-1}(N)$ is a neighborhood of a. Then for any neighborhood M of $f(a)$, M contains an element $B \in \mathcal{B}_{f(a)}$, which is a neighborhood of $f(a)$. Hence, $f^{-1}(M)$ contains $f^{-1}(B)$, a neighborhood of a, which implies that $f^{-1}(M)$ is a neighborhood of a. Thus, f is continuous.

2.4.4

(i) Observe that $\bigcup\limits_{\epsilon>0} [a-\epsilon, a+\epsilon] \supseteq \bigcup\limits_{\epsilon>0} (a-\epsilon, a+\epsilon)$. Therefore, for any neighborhood N of a, N contains, for some $\epsilon_0 > 0$, the interval $B(a;\epsilon_0) \subset \bigcup\limits_{\epsilon>0} (a-\epsilon, a+\epsilon) \subseteq \bigcup\limits_{\epsilon>0} [a-\epsilon, a+\epsilon] \Rightarrow \bigcup\limits_{\epsilon>0} [a-\epsilon, a+\epsilon]$ is a basis for the system of neighborhoods at a.

(ii) Let $\mathcal{B}_a := \{B(a;,\epsilon) : \epsilon > 0 \wedge \epsilon \in \mathbb{Q}\}$. Then for any neighborhood N of a, N contains, for some $\epsilon_0 > 0$, the interval $B(a;\epsilon_0)$. Since $\epsilon_0 > 0$, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $\epsilon_1 > 0$ so that $\epsilon_1 < \epsilon_0$. Hence, $N \supset B(a;\epsilon_1)$ and $B(a;\epsilon_1) \in \mathcal{B}_a \Rightarrow \mathcal{B}_a$ is a basis for the system of neighborhoods at a.

(iii) Let $\mathcal{B}_a := \{B(a;\frac{1}{n}) : n \in \mathbb{N}\}$. Then for any neighborhood N of a, N contains, for some $\epsilon_0 > 0$, the interval $B(a;\epsilon_0)$. Since the sequence $\{\frac{1}{n}\}$ converges to 0, there exists $n_0 \in \mathbb{N}$ so that $n \geq n_0 \Rightarrow \frac{1}{n} < \epsilon_0$. Hence, for any $n \geq n_0$, $N \supset B(a;\frac{1}{n})$ and $B(a;n) \in \mathcal{B}_a \Rightarrow \mathcal{B}_a$ is a basis for the system of neighborhoods at a.

(iv) The reasoning in this subproblem is analogous to that in the previous subproblem (iii). The only difference is that in this subproblem we require that $n \geq \max\{n_0, k\}$.

Now, assume for the sake of contradiction that in $\mathbb{R}$ there exists a finite collection of sets $\tilde{\mathcal{B}_a}$ which forms a basis for the system of neighborhoods at a. Since $\tilde{\mathcal{B}_a}$ is finite, we may explicitly list its elements: suppose $\tilde{\mathcal{B}_a} = \{B_1, B_2, ..., B_n\}$. Let $B := \bigcap\limits_{i=1}^{n} B_i$. Then B is a neighborhood of a and $B \subseteq B_i$ for $1 \leq i \leq n$. Moreover, there exists $\delta > 0$ such that the real interval $B(a;\delta) = (a-\delta, a+\delta) \subseteq B$. Now, let $\delta^* := \frac{\delta}{2}$; then $N := B(a;\delta^*) \subsetneq B$ is a neighborhood of a and there does not exists a B_i, $1 \leq i \leq n$, so that $B_i \subseteq B(a;\delta^*)$. Thus, we have a contradiction. Consequentially, there does not exist a finite collection of subsets of $\mathbb{R}$ that can be a basis for the system of neighborhoods of a.

2.4.5

Given $a \in X$, we want to show that there exists a collection of neighborhoods $\{B_n\}_{n\in\mathbb{N}}$ which constitutes a basis for the system of neighborhoods at a. Let $B_n := B(a; \frac{1}{n})$. Then for any neighborhood N of a, there exists $\epsilon > 0$ such that $N \supseteq B(a;\epsilon)$. Since $\frac{1}{n} \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ so that for any $n \geq n_0$, $\frac{1}{n} < \epsilon$, which implies that for any $n \geq n_0$, $N \supset B(a; \frac{1}{n}) = B_n \Rightarrow \{B_n\}_{n\in\mathbb{N}}$ constitutes a basis for the system of neighborhoods at a.

2.4.6

$a, b \in X$ such that $a \neq b \Rightarrow d(a,b) > 0$; suppose $d(a,b) = \delta$. Then let $N_a := B(a; \frac{\delta}{2})$ and $N_b := B(a; \frac{\delta}{2})$. I claim that $N_a \cap N_b$. To prove this claim, it suffices to show that $d(a,x) < \frac{\delta}{2} \Rightarrow d(b,x) > \frac{\delta}{2}$ (because this is equivalent to proving that $x \in N_a \Rightarrow x \notin N_b$, and by symmetry we may conclude that $x \in N_b \Rightarrow x \notin N_a$).

Observe that $d(a,x) < \frac{\delta}{2}$ implies that:

$$\begin{aligned} d(a,b) \leq d(a,x) + d(x,b) \iff \ & \delta - d(a,x) \leq d(x,b) \\ & \Rightarrow \delta - \frac{\delta}{2} < \delta - d(a,x) \leq d(x,b) \\ & \Rightarrow d(x,b) > \frac{\delta}{2} \end{aligned}$$

2.4.7

$a \in X$ is a point $a = (a_1, ..., a_n)$ where $a_i \in X_i$ for $i = 1, ..., n$. Let $B(a;\delta) \subset X$. Then,

$$\begin{aligned} B(a;\delta) &= \{x \in X : d(a,x) < \delta\} \\ &= \{(x_1, ..., x_n) : x_i \in X_i \ \forall i \in [n] \ \wedge \ \max_{1\leq i\leq n}\{d_i(a_i, x_i)\} < \delta\} \\ &= \{(x_1, ..., x_n) : x_i \in X_i \ \forall i \in [n] \ \wedge \ d_i(a_i, x_i) < \delta \ \forall i \in [n]\} \\ &= \prod_{i=1}^{n} \{x_i \in X_i : d_i(a_i, x_i) < \delta\} \\ &= \prod_{i=1}^{n} B_i(a_i;\delta) \end{aligned}$$

Given that $\mathcal{B}_{a_i}$ is a basis for the system of neighborhoods at a_i, and that $\mathcal{B}_a = \bigcup_{B_i \in \mathcal{B}_{a_i}} \prod_{i=1}^{n} B_i$, we want to show that $\mathcal{B}_a$ is a basis for the system of neighborhoods at a. Suppose $N \subset X$ is a neighborhood of a. Then there exists $\delta > 0$ such that $N \supseteq B(a;\delta) = \prod_{i=1}^{n} B_i(a_i;\delta)$. For each $i \in [n]$, $B_i(a_i;\delta)$ is a neighborhood of $X_i \Rightarrow$ for each $i \in [n]$, $B_i(a_i;\delta) \supseteq B_i$ for some $B_i \in \mathcal{B}_{a_i} \Rightarrow B(a;\delta) \supseteq \prod_{i=1}^{n} B_i$ where $B_i \in \mathcal{B}_{a_i}$ $\forall i \in [n] \Rightarrow B(a;\delta) \supseteq B = \prod_{i=1}^{n} B_i \in \mathcal{B}_a$. Hence, $\mathcal{B}_a$ is a basis for the system of neighborhoods of a.

Now, for each $i \in [n]$, let $p_i : X \to X_i$ be the projection that maps $p_i(a) = a_i$. We want to show that for each $i \in [n]$, p_i is continuous; i.e., we want to show that for every neighborhood M of $p_i(a)$, $p_i^{-1}(M)$ is a neighborhood of a. Let N_i be a neighborhood of $p_i(a) = a_i$. Then $N_i \supseteq B_i(a_i;\delta) \Rightarrow p_i^{-1}(N_i) \supseteq p_i^{-1}\big(B_i(a_i;\delta)\big) = \{(x_1, ..., x_n) : x_i \in X_i \ \forall i \in [n] \ \wedge \ d_i(a_i, x_i) < \delta\} \supset B(a;\delta) \Rightarrow p_i$ is continuous.

Now, suppose $f : Y \to X$ is a continuous function. Then since for each $i \in [n]$, p_i is continuous, it follows immediately that $p_i \circ f$ is continuous. Conversely, suppose for each $i \in [n]$, $p_i \circ f$ is continuous. Then given $b \in Y$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $(p_i \circ f)(B(b;\delta)) \subseteq B\big((p_i \circ f)(b);\epsilon\big)$, for every $i = 1, ..., n$. Given $b \in Y$, $f(b) = a$ for some $a \in X$; consequentially, for each $i \in [n]$, $(p_i \circ f)(b) = p_i\big(f(b)\big) = p_i(a) = a_i \Rightarrow \forall i \in [n]$, $B(a_i;\epsilon) = B\big((p_i \circ f)(b);\epsilon\big) \supseteq (p_i \circ f)\big(B(b;\delta)\big)$. Hence, $B(b;\delta) = \{y \in Y : d(b,y) < \delta\} \subseteq \{y \in Y : f(y) = x \ \wedge \ d(x,a) < \epsilon\} \Rightarrow f\big(B(b;\delta)\big) \subseteq B\big(f(b);\epsilon\big)$. Thus, $f : Y \to X$ is continuous.

2.4.8

We are told that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that there exists $a \in \mathbb{R}$ such that $f(a) > 0$. We want to show that there exists $k > 0$ and a closed interval $F = [a - \delta, a + \delta]$ such that $f(x) \geq k \; \forall x \in F$.

Recall that f is continuous at a iff $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $f\big(B(a;\delta)\big) \subset B\big(f(a);\epsilon\big)$. Now, $f(a) > 0 \Rightarrow \exists\; k > 0$ such that $f(a) > k > 0$ by the density of $\mathbb{R}$. Choose $\epsilon > 0$ so that $\epsilon < f(a) - k$. Then by continuity of f at a, there exists $\delta_\epsilon > 0$ such that $f\big(B(a;\delta_\epsilon)\big) \subset B\big(f(a);\epsilon\big) \iff \exists\, \delta_\epsilon > 0$ such that $f\big((a - \delta_\epsilon, a + \delta_\epsilon)\big) \subset \big(f(a) - \epsilon, f(a) + \epsilon\big) = \big(k, 2f(a) - k\big) \Rightarrow f(x) \geq k \;\forall\, x \in (a - \delta_\epsilon, a + \delta_\epsilon)$. Choose $\delta > 0$ so that $\delta < \delta_\epsilon$, and set $F := [a - \delta, a + \delta]$. Then $f(x) \geq k \;\forall\, x \in F$.

2.5

2.5.1

We are given the metric space $\left(\prod_{i=1}^{k} X_i, d\right)$ where $d(x,y) = \max_{1 \leq i \leq k}\{d_i(x_i, y_i)\}$. $a_1, a_2, \ldots$ are points in X where $a_n = (a_1^n, a_2^n, \ldots, a_k^n)$ and $c = (c_1, c_2, \ldots, c_k) \in X$. We want to show that $\lim_{n\to\infty} a_n = c \iff \lim_{n\to\infty} a_i^n = c_i$ for each $i \in [k]$.

($\Rightarrow$) Suppose $\lim_{n\to\infty} a_n = c$. Then for every neighborhood V of c, there exists $N \in \mathbb{N}$ such that $a_n \in V$ for $n \geq N$. Therefore, for any $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $a_n \in B(c; \frac{1}{m})$ for $n \geq N$; this implies that for any $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ so that for any $n \geq N$, $d(a_n, c) < \frac{1}{m} \iff \max_{1\leq i \leq k}\{d_i(a_i^n, c_i)\} < \frac{1}{m} \Rightarrow \lim_{n\to\infty} d(a_i^n, c_i) = 0$ for $i = 1, 2, \ldots, k \Rightarrow \lim_{n\to\infty} a_i^n = c_i$.

($\Leftarrow$) Suppose that for $i = 1, 2, \ldots, k$, $\lim_{n\to\infty} a_i^n = c_i$. Then for $i = 1, 2, \ldots, k$, for every neighborhood V_i of c_i, there exists $N_i \in \mathbb{N}$ such that $a_i^n \in V_i$ for $n \geq N_i \Rightarrow$ for any $m \in \mathbb{N}$, there exists N_i (for $i = 1, 2, \ldots, k$) such that $a_i^n \in B(c_i, \frac{1}{m})$ for $n \geq N_i \Rightarrow$ for $i = 1, 2, \ldots, k$, $d_i(a_i^n, c_i) < \frac{1}{m} \Rightarrow \max_{1\leq i\leq k}\{d_i(a_i^n, c_i)\} < \frac{1}{m} \Rightarrow \lim_{n\to\infty} d(a_n, c) = 0 \Rightarrow \lim_{n\to\infty} a_n = c$.

2.5.2

Recall that $d(x,y) = \max_{1\leq i\leq k}\{|x_i - y_i|\}$, $d'(x,y) = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$, and $d''(x,y) = \sum_{i=1}^{k}|x_i - y_i|$; also, recall from exercise 2.2.2, $d'(x,y) \leq \sqrt{n}\cdot d(x,y)$ and $d''(x,y) \leq n \cdot d(x,y)$.

Therefore, if $\{a_i\}_{i\in\mathbb{N}}$ is a sequence in $\mathbb{R}^k$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} d(a_n, a) = 0 \Rightarrow \lim_{n\to\infty} d'(a_n, a) \leq \sqrt{k}\cdot \lim_{n\to\infty} d(a_n, a) = \sqrt{k}\cdot 0 = 0$ and $\lim_{n\to\infty} d''(a_n, a) \leq k \cdot \lim_{n\to\infty} d(a_n, a) = k \cdot 0 = 0$. Therefore, $\lim_{n\to\infty} d(a_n, a) = 0 \Rightarrow \lim_{n\to\infty} d'(a_n, a) = 0$ and $\lim_{n\to\infty} d''(a_n, a) = 0$. Moreover, from exercise 2.2.2, $d(x,y) \leq d'(x,y)$ and $d(x,y) \leq d''(x,y)$; therefore, $d'(a_n, a) = 0$ or $d''(a_n, a) = 0$ implies that $d(a_n, a) = 0$. Thus, $\lim_{n\to\infty} d(a_n, a) = 0 \iff \lim_{n\to\infty} d'(a_n, a) = 0 \iff \lim_{n\to\infty} d''(a_n, a) = 0$.

2.5.3

Suppose the sequence $\{a_n\}_{n\in\mathbb{N}}$ of points in the metric space (X, d) converges to the point a. Then $\lim_{n\to\infty} d(a_n, a) = 0 \iff \forall \epsilon, \exists\, N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N$. If $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}$, then recall that n_k is a strictly increasing sequence from $\mathbb{N}$ to $\mathbb{N}$; thus, $d(a_{n_k}, a) < \epsilon$ for all $n_k \geq N \Rightarrow \lim_{k\to\infty} d(a_{n_k}, a) = 0 \iff \lim_{k\to\infty} a_{n_k} = a$.

2.5.4

Let $\{a_i\}_{i\in\mathbb{N}}$ be a convergent sequence of real numbers that converges to $a \in \mathbb{R}$. We want to show that $\{a_i\}$ is bounded; i.e., we want to show that $\forall\ \epsilon > 0$, $\exists\ M > 0$ such that $|a_i| \leq M\ \forall\ i \in \mathbb{N}$. $\{a_i\}$ converges to $a \Rightarrow \forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$. $|a_n - a| < \epsilon \Rightarrow |a_n| < |a| + \epsilon$; therefore, given $\epsilon > 0$, let $M = \max\{|a_1|, |a_2|, ..., |a_{n-1}|, |a| + \epsilon\}$. Then $M \geq |a_i|\ \forall\ i \in \mathbb{N}$; i.e., $\{a_i\}$ is bounded.

Let $\{a_i\}_{i\in\mathbb{N}}$ be a non-decreasing sequence bounded above. Then for every $i \in \mathbb{N}$, $a_i \leq a_{i+1}$ and there exists $M \in \mathbb{R}$ such that $a_i \leq M$. Since $\{a_i\} \subset \mathbb{R}$ is bounded above, by the completeness axiom, there exists a least upper bound of $\{a_i\}$, which we denote as a. $a =$ l.u.b.$\{a_i\}$ implies that $\forall\ \epsilon > 0$, $\exists\ a_n \in \{a_i\}$ such that $a - a_n < \epsilon \Rightarrow \forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $|a - a_n| < \epsilon$ for $n \geq N \Rightarrow \{a_i\}$ converges to a. The proof showing that a non-increasing sequence bounded below converges to its greatest lower bound is analogous.

2.5.5

Omitted.

2.5.6

Recall that $d(x, A) =$ g.l.b.$\{d(x, a) : a \in A\}$ and $d(y, A) =$ g.l.b.$\{d(y, a) : a \in A\}$. We want to show that $d(x, A) \leq d(x, y) + d(y, A)$. Consider the following cases:

- Suppose $x \in A$. Then $d(x, A) = 0 \leq d(x, y) + d(y, A)$
- Suppose $x \notin A$ but $y \in A$. Then $d(x, A) =$ g.l.b.$\{d(x, a) : a \in A\} \leq d(x, y) \Rightarrow d(x, A) \leq d(x, y) + d(y, A)$
- Suppose $x, y \notin A$. Then there exists $x' \in A$ and $y' \in A$ such that $d(x, A) = d(x, x')$ and $d(y, A) = d(y, y')$. Hence,
$$d(x, A) = d(x, x') \leq d(x, y) + d(y, y') + d(y', x') = d(x, y) + d(y, A)$$

Therefore, after exhausting all cases, we have $d(x, A) \leq d(x, y) + d(y, A)$.

2.5.7

Given a nonempty subset A of the metric space (X, d), we want to prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous; i.e., we want to show that $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Recall from the previous exercise, for any $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A) \Rightarrow d(x, A) - d(y, A) \leq d(x, y)$; since x and y are arbitrary, we also have: $d(y, A) - d(x, A) \leq d(x, y)$. Therefore, $|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y)$; thus, given $\epsilon > 0$, letting $\delta := \epsilon$ we have $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. That is, f is continuous.

2.5.8

Give a nonempty subset A of the metric space (X, d) and a point $x \in X$, we want to show that $d(x, A) = 0$ if and only if every neighborhood of x contains a point $y \in A$.

$(\Rightarrow)$ $d(x, A) = 0 \iff$ glb$\{d(x, A)\} = 0 \Rightarrow \forall\ \epsilon > 0$, $\exists\ y \in A$ such that $d(x, y) < \epsilon$. Given a neighborhood M of x, $\exists\ \delta > 0$ such that $B(a; \delta) \subset M$; therefore, for any neighborhood M of a, $\exists\ y \in A$ such that $y \in B(a; \delta) \subset M$.

$(\Leftarrow)$ Suppose every neighborhood M of x contains a point $y \in A$. Then for every $n \in \mathbb{N}$, there exists $y \in A$ such that $y \in B(a; \frac{1}{n})$. Since $\forall\ \epsilon > 0$, $\exists\ m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$, this implies that $\forall \epsilon > 0$, $\exists\ y \in A$ such that $d(x, y) < \epsilon \Rightarrow d(x, A) = 0$.

2.5.9

Omitted.

2.6

2.6.1

Given that (X_i, d_i), $i = 1, 2, ..., n$, are metric spaces, we form the set $X = \prod_{i=1}^{n} X_i$ equipped with the metric $d : X \times X \to \mathbb{R}_{\geq 0}$ defined as $d(x, y) = \max_{1 \leq i \leq n} \{d_i(x_i, y_i)\}$. We want to prove that for $i = 1, 2, ..., n$, if $\mathcal{O}_i$ is an open subset of X_i, then $\bigtimes_{i=1}^{n} \mathcal{O}_i$ is an open subset of X.

$\bigtimes_{i=1}^{n} = \{(x_1, x_2, ..., x_n) : x_i \in \mathcal{O}_i, i = 1, 2, ..., n\}$. Since $x_i \in \mathcal{O}_i$, $\exists\ \delta_i > 0$ such that $B(x_i; \delta_i) \subset \mathcal{O}_i$ for $i = 1, 2, ..., n$. Let $\delta := \frac{1}{2} \min_{1 \leq i \leq n} \{\delta_i\}$. Then, $\delta > 0$ and $B(x_i; \delta) \subset \mathcal{O}_i$ for $i = 1, 2, ..., n \Rightarrow B(x; \delta) = B\big((x_1, x_2, ..., x_n); \delta\big) \subset \mathcal{O}_1 \times \mathcal{O}_2 \times ... \times \mathcal{O}_n \Rightarrow \bigtimes_{i=1}^{n} \mathcal{O}_i$ is open.

Now, suppose $\mathcal{O}$ is an open subset of X; we want to show that $\mathcal{O} = \bigcup_{\alpha \in I} \big(\bigtimes_{i=1}^{n} O_i^{\alpha}\big)$, where $\mathcal{O}_i^{\alpha}$ are open sets for $i = 1, 2..., n$ and every $\alpha \in I$. Since $\mathcal{O} \subset X$, there are sets $A_i \subset X_i$, $i = 1, 2, ..., n$, such that $\mathcal{O} = A_1 \times A_2 \times ... \times A_n$. If $x = (x_1, x_2, ..., x_n) \in \mathcal{O}$, then there exists $\delta > 0$ so that $B(x; \delta) \subset \mathcal{O}$. Hence, for $i = 1, 2, ..., n$, $B(x_i; \delta) \subset A_i \Rightarrow$ each set A_i, $i = 1, 2, ..., n$, is open. Since the arbitrary union of open sets is open, this implies that there exists open subsets $\mathcal{O}_i^{\alpha}$, $\alpha \in I$ and $i = 1, 2, ..., n$, such that $\mathcal{O} = \bigcup_{\alpha \in I} \big(\bigtimes_{i=1}^{n} \mathcal{O}_i^{\alpha}\big)$.

2.6.2

Given the metric space (X, d) with metric $d : X \times X \to \mathbb{R}_{\geq 0}$ defined as

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

we want to prove that every subset of X is open.

Let $A \subset X$. Then if $a \in A$, $\forall\, x \in X \setminus \{a\}$, $d(a, x) = 1 \Rightarrow B(a; \frac{1}{2}) = \{a\} \subset A \Rightarrow A$ is open. QED.

2.6.3

We are told that (X, d_1) and (Y, d_2) are metric spaces, and we form the metric space $(X \times Y, d)$ where $d : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ is defined as $d(a, b) = \max_{1 \leq i \leq 2} \{d_i(a_i, b_i)\}$. Given that $f : X \to Y$ is continuous, we want to show that the graph of f, $\Gamma_f = \{(x, f(x)) : x \in X\}$, is closed.

Let $\{(x_n, f(x_n))\}_{n=1}^{\infty}$ be a sequence of points in Γ_f which converges to the point $(x, y) \in X \times Y$. Then $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} f(x_n) = y$. $x \in X$, hence by continuity of f, $\lim_{n \to \infty} f(x_n) = f(x) \Rightarrow f(x) = y \Rightarrow (x, y) \in \Gamma_f$. Thus, Γ_f is closed.

2.6.4

We are told that $f : \mathbb{R} \to \mathbb{R}$ is defined as:

$$f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x \neq 0 \end{cases}$$

and we want to show that Γ_f is a closed subset of $(\mathbb{R}^2, d)$, but that f is not continuous.

$f|_{(-\infty,0)}(x) = 0$ and $f|_{(0,\infty)}(x) = \frac{1}{x}$, thus it is clear that f is continuous on $(-\infty, 0)$ and $(0, \infty) \Rightarrow \Gamma_{f|_{(-\infty,0]}}$ and $\Gamma_{f|_{(0,\infty)}}$ are closed. Moreover, $\{0, f(0)\} = \{(0, 0)\} \Rightarrow C_{\mathbb{R}^2}((0, f(0)) = \big((-\infty, 0) \cup (0, \infty), (-\infty, 0) \cup (0, \infty)\big) = \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\}$, which is open since the finite product of open sets is also open. Hence, $\{0, f(0)\}$ is closed. Since the finite union of closed sets is closed, $\Gamma_{f|_{(-\infty,0)}} \cup \{(0, f(0))\} \cup \Gamma_{f|_{(0,\infty)}} = \Gamma_f$ is closed.

Note, however, that f is not continuous; in particular, f is discontinuous at $x = 0$. Observe that the sequence $\{x_n = \frac{1}{n}\}_{n=1}^{\infty}$ converges to 0, but, $f(x_n) = f(\frac{1}{n}) = \frac{1}{\frac{1}{n}} = n \to \infty \neq f(0) = 0 \Rightarrow f$ is not continuous at 0.

2.6.5

We are told that A is a non-empty, closed subset of $\mathbb{R}$, and that A is bounded below. By the completeness axiom, there exists a greatest lower bound of A, $\alpha \in \mathbb{R}$, where $\alpha \leq a \; \forall a \in A$. We want to show that $A \ni \alpha$.

Since α is the greateast lower bound of A, for any $\epsilon > 0$, there exists $a \in A$ such that $a < \alpha + \epsilon$. Hence, $\forall \; n \in \mathbb{N}$, $\exists \; a_n \in A$ such that $a_n < \alpha + \frac{1}{n} \Rightarrow a_n \xrightarrow{n \to \infty} \alpha$. A closed $\iff$ any sequence in A which converges to a point $x \in \mathbb{R}$ implies that $x \in A$; thus, $A \ni \alpha$.

2.6.6

Recall that $A' = \{x \in X : \forall \; \epsilon > 0, \exists \; y \in A \text{ s.t. } y \neq x \wedge y \in B(x; \epsilon)\}$, and $A^i = \{a \in A : \exists \; \delta > 0 \text{ s.t. } B(a; \delta) \cap A = a\}$. Thus, it immediately follows that $A' \cap A^i = \emptyset$. If $x \in A$, then given any $\epsilon > 0$, either $B(x; \epsilon) \subseteq \{x\}$, or there exists $y \in A$ such that $B(x; \epsilon) \supseteq \{x, y\}$. If no such y exists for any $\epsilon > 0$, then $x \in A^i$, otherwise $x \in A'$. That is, $A \subseteq A' \cup A^i$.

Now, let $\overline{A} = A' \cup A^i$. Then we want to show that $x \in \overline{A}$ if and only if there exists $\{a_n\}_{n=1}^{\infty} \subset A$ such that $a_n \xrightarrow{n \to \infty} x$. So, suppose $x \in \overline{A}$. Then $x \in A'$ or $x \in A^i$, but $x \notin A' \cap A^i$. If $x \in A'$ then $\forall \; \epsilon > 0$, there exists $y \in A$ such that $y \neq x$ and $y \in B(x; \epsilon)$; equivalently, $\forall \; n \in \mathbb{N}$, $\exists \; A \ni y := a_n$ such that $a_n \neq x$ and $a_n \in B(x; \frac{1}{n}) \Rightarrow$ there exists $\{a_n\} \subset A$ such that $\lim\limits_{n \to \infty} a_n = x$. Alternatively, if $x \in A^i$, then $\exists \; \delta > 0$ such that $B(x; \delta) \cap A = \{x\}$. Observe that the sequence $\{x\}_{n=1}^{\infty} = x, x, ... \subset A$ and $\lim\limits_{n \to \infty} x = x$. Conversley, if $\{a_n\}_{n=1}^{\infty} \subset A$ such that $\lim\limits_{n \to \infty} a_n = x$, then $\forall \; \epsilon > 0$, $\exists \; N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n \in B(x; \epsilon) \Rightarrow x \in A' \subset \overline{A}$.

Now, let F be a closed set such that $F \supset A$. Then F closed $\iff$ F contains all of its limits points. Since $A \subset F$, this implies that F contains all limit points of $A \Rightarrow A' \subset F$. Furthermore, since $A^i \subset A \subset F$, this implies that $F \supset A^i$. Hence, $\overline{A} = A' \cup A^i \subset F$. Since F is an arbitrary closed set containing A, and $\overline{A} \subset F$, this implies that $\overline{A} \subseteq \bigcap\limits_{F \supset A, \, F \text{ closed}} F$; moreover, since $\overline{A} \supset A$ and $\overline{A}$ is closed (since we showed that $\overline{A}$ contains all limit points of A), this implies that $\overline{A} \supseteq \bigcap\limits_{F \supset A, \, F \text{ closed}} F$; thus, $\overline{A} = \bigcap\limits_{F \supset A, \, F \text{ closed}} F$.

2.7

2.7.1

Given $a, b \in \mathbb{R}^n$, define the functions $f : \mathbb{R}^n \to \mathbb{R}^n$ as $f(x) = f\big((x_1, ..., x_n)\big) = (x_1 + b_1 - a_1, ..., x_n + b_n - a_n)$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ as $g(x) = g\big((x_1, ..., x_n)\big) = (x_1 - b_1 + a_1, ..., x_n - b_n + a_n)$. Then observe that $f(a) = f\big(a_1, ..., a_n\big) = (a_1 + b_1 - a_1, ..., a_n + b_n - a_n) = (b_1, ..., b_n) = b$. Moreover, we have:

$$
\begin{aligned}
(f \circ g)(x) &= f\big(g(x_1, ..., x_n)\big) \\
&= f(x_1 - b_1 + a_1, ..., x_n - b_n + a_n) \\
&= \big((x_1 - b_1 + a_n) + b_1 - a_1, ..., (x_n - b_n + a_n) + b_n - a_n\big) \\
&= (x_1, ..., x_n)
\end{aligned}
$$

$\Rightarrow f \circ g = id_{\mathbb{R}^n}$. Similarly, we have:

$$
\begin{aligned}
(g \circ f)(x) &= g\big(f(x_1, ..., x_n)\big) \\
&= f(x_1 + b_1 - a_1, ..., x_n + b_n - a_n) \\
&= \big((x_1 + b_1 - a_n) - b_1 + a_1, ..., (x_n + b_n - a_n) - b_n + a_n\big) \\
&= (x_1, ..., x_n)
\end{aligned}
$$

$\Rightarrow g \circ f = id_{\mathbb{R}^n}$. Thus, f and g are inverses.

Now, we want to show that f and g are continuous. Given $\epsilon > 0$, let $\delta = \epsilon$. Then observe that for any $x, y \in \mathbb{R}^n$, when $d(x, y) < \delta$ we have:

$$\begin{aligned} d\big(f(x), f(y)\big) &= \max_{1 \le i \le n} \{|(x_i + b_i - a_i) - (y_i + b_i - a_i)|\} \\ &= \max_{1 \le i \le n} \{|x_i - y_i|\} \\ &= d(x, y) \\ &< \delta = \epsilon \end{aligned}$$

$\Rightarrow f$ is continuous. The proof that g is continuous is analogous. Therefore, we conclude that there is an equivalence between $\mathbb{R}^n$ and itself such that $f(a) = b$.

2.7.2

Let $\phi(x) = \tan(x)$. Then $\phi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is continuous, one-to-one and onto, with inverse function $\phi^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined as $\phi^{-1}(x) = \arctan(x)$, which is also continuous, one-to-one, and onto. Therefore, $(-\frac{\pi}{2}, \frac{\pi}{2})$ is topologically equivalent to $\mathbb{R}$.

Now, we want to show that any two open intervals, considered as subspaces of the real number system, are topologically equivalent. To do so, we first establish the following lemma:

Lemma 1. *If (X, d_X) and (Y, d_Y) are both topologically equivalent to (Z, d_Z), then (X, d_X) and (Y, d_Y) are topologically equivalent.*

Proof. (X, d_X) topologically equivalent to (Z, d_Z) means that there exists a continuous inverse functions $f : X \to Z$ and $f^{-1} : Z \to X$, and similarly (Y, d_Y) topologically equivalent to (Z, d_Z) means that there exists continuous inverse functions $g : Y \to Z$ and $g^{-1} : Z \to Y$. The composition of bijective functions is a bijection, and the composition of continuous functions is a continuous function; therefore, $g^{-1} \circ f : X \to Y$ is a continuous bijection, as well as its inverse $f^{-1} \circ g : Y \to X$. Hence, (X, d_X) is topologically equivalent to (Y, d_Y). ☐

Let $X \subset \mathbb{R}$ be an open interval. Define $h := \phi^{-1}|_X$. Then h is continuous, one-to-one, and onto; moreover, there exists inverse function $h^{-1} := \phi|_{\phi^{-1}(X)}$, which is also continuous, one-to-one, and onto. Therefore, X and $(-\frac{\pi}{2}, \frac{\pi}{2})$ are topologically equivalent. Analogously, if $Y \subset \mathbb{R}$ is an open interveal, then similar reasoning implies that Y and $(-\frac{\pi}{2}, \frac{\pi}{2})$ are topoligically equivalent. Therefore, by the above lemma, we conclude that X and Y are topologically equivalent. Furthermore, we also conclude that any open interval is topologically equivalent to $\mathbb{R}$.

2.7.3

We are told that for $i = 1, 2, ..., n$, (X_i, d_i) is topologically equivalent to (Y_i, d'_i); that is, for $i = 1, ..., n$, there exists continuous inverse functions $f_i : X_i \to Y_i$ and $f_i^{-1} : Y_i \to X_i$. $X := \prod_{i=1}^{n} X_i$ is equipped with the metric $d_X : X \times X \to \mathbb{R}_{\ge 0}$ defined as $d_X(x, y) = d_X\big((x_1, ..., x_n), (y_1, ..., y_n)\big) = \max_{1 \le i \le n} \{d_i(x_i, y_i)\}$, and $Y := \prod_{i=1}^{n} Y_i$ is equipped with the metric $d_Y : Y \times Y \to \mathbb{R}_{\ge 0}$ is defined as $d_Y(x, y) = d_Y\big(x_1, ..., x_n), (y_1, ..., y_n)\big) = \max_{1 \le i \le n} \{d' -_i (x_i, y_i)\}$. We want to show that X and Y are topologically equivalent.

Define $f : X \to Y$ as $f(x) = f\big((x_1, ..., x_n)\big) = \big(f_1(x_1), ..., f_n(x_n)\big)$. Then, since each f_i is bijective and has an inverse, we can define $g : Y \to X$ as $g(y) = g\big((y_1, ..., y_n)\big) = \big((f_1^{-1}(y_1), ..., f_n^{-1}(y_n)\big)$. Then observe that

$$\begin{aligned} (f \circ g)(y) &= f\big(g(y_1, ..., y_n)\big) \\ &= f\big((f_1^{-1}(y_1), ..., f_n^{-1}(y_n)\big) \\ &= \big(f_1(f_1^{-1}(y_1)), ..., f_n(f^{-1}(y_n)\big) \\ &= (y_1, ..., y_n) \end{aligned}$$

$\Rightarrow f \circ g = id_Y$. Similarly,

$$\begin{aligned}(g \circ f)(x) &= g\big(f(x_1, ..., x_n)\big) \\ &= g\big((f_1(x_1), ..., f_n(x_n)\big) \\ &= \big(f_1^{-1}(f_1(x_1)), ..., f_n^{-1}(f(x_n)\big) \\ &= (x_1, ..., x_n)\end{aligned}$$

$\Rightarrow g \circ f = id_X$. Hence, f and g are inverses; i.e., $g = f^{-1}$.

Now, since for $i = 1, ..., n$, each $f_i : X_i \to Y_i$ is continuous, this means that $\forall\, \epsilon > 0$, $\exists\, \delta_i > 0$ such that for all $x_i, y_i \in X_i$, $d_i(x_i, y_i) < \delta_i$ implies that $d_i'\big(f_i(x_i), f_i(y_i)\big) < \epsilon$. Since there are only finitley many δ_i, define $\delta := \max_{1 \le i \le n}\{d_i(x_i, y_i)\}$. Then, given $\epsilon > 0$, for $x, y \in X$, $d_X(x, y) = d_X\big((x_1, ..., x_n), (y_1, ..., y_n)\big) < \delta$ implies that

$$\begin{aligned}d_Y\big(f(x), f(y)\big) &= d_Y\big((f_1(x_1), ..., f_n(x_n)), (f_1(y_1), ..., f_n(y_n))\big) \\ &= \max_{1 \le i \le n}\{d_i'\big(f_i(x_i), f_i(y_i)\big)\} \\ &< \epsilon\end{aligned}$$

$\Rightarrow f$ is continuous. Thus, X and Y are topologically equivalent.

2.7.4

Let $X_i = (0, 1) \subset \mathbb{R}$. Then by exercise 2.7.2, we know that X_i is topologically equivalent to $\mathbb{R}$. Now, by exercise 2.7.3, we conclude that $\prod_{i=1}^{n} X_i = \{(x_1, ..., x_n) \in \mathbb{R}^n : 0 < x_i < 1, i = 1, ..., n\}$ is topologically equivalent to $\prod_{i=1}^{n} \mathbb{R} = \mathbb{R}^n$.

2.7.5

We want to show that metric equivalence, or isometry, is an equivalence relation.

1. Given a metric space (X, d), the identity function $id : X \to X$ is a bijection and $\forall\, x, y \in X$, $d\big(id(x), id(y)\big) = d(x, y) \Rightarrow XRX$.

2. Suppose (X, d_X) and (Y, d_Y) are metric spaces, and XRY. Then there exists bijective function $f : X \to Y$ such that $d_Y\big(f(x_1), f(x_2)\big) = d_X(x_1, x_2)$. Since f is a bijection, it has an inverse function $f^{-1} : Y \to X$, which is also a bijection; thus, if $y_1, y_2 \in Y$, then there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Therefore,
$$d_Y(y_1, y_2) = d_Y\big(f(x_1), f(x_2)\big) = d_X(x_1, x_2) = d_X\big(f^{-1}(y_1)f^{-1}(y_2)\big)$$
Thus, YRX.

3. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces, and suppose XRZ and YRZ. Then there exist bijections $f : X \to Z$ and $g : Y \to Z$ such that $d_X(x_1, x_2) = d_Z\big(f(x_1), f(x_2)\big)$ and $d_Y(y_1, y_2) = d_Z\big(g(y_1), g(y_2)\big)$, for any $x_1, x_2 \in X$ and for any $y_1, y_2 \in Y$. Then there exists an inverse function $g^{-1} : Z \to Y$ such that $d_Z(z_1, z_2) = d_Y\big(g^{-1}(z_1), g^{-1}(z_2)\big)\ \forall\, z_1, z_2 \in Z$; furthermore, $g^{-1} \circ f : X \to Y$ is a bijective function (since it is the composition of bijections). Therefore, if $x_1, x_2 \in X$ and $f(x_1) = z_1$, $f(x_2) = z_2$, for some $z_1, z_2 \in Z$, then
$$\begin{aligned}d_X(x_1, x_2) &= d_Z\big(f(x_1), f(x_2)\big) \\ &= d_Z(z_1, z_2) \\ &= d_Y\big(g^{-1}(f(x_1)), g^{-1}(f(x_2))\big) \\ &= d_Y\big((g^{-1} \circ f)(x_1), (g^{-1} \circ f)(x_2)\big)\end{aligned}$$
$\Rightarrow XRY$.

Thus, metric equivalence is an equivalence relation. Now, since metric equivalence implies topological equivalence, we conclude that topological equivalence is also an equivalence relation.

2.7.6

We are told that (Y, d') be a subspace of the metric space (X, d). First we want to show that subset O' of Y is open $\iff$ there exists an open subset O of X such that $O' = Y \cap O$.

($\Rightarrow$) Suppose $O' \subset Y$ is open. Since $X \supset Y$, this implies that $X \supset O'$. Therefore, there exists an open subset O of X such that $O' = Y \cap O$; namely, O'.

($\Leftarrow$) Let O be an open subset of X and suppose $O' = Y \cap O$. Then since Y is a metric space, Y is open $\Rightarrow O' = Y \cap O$ is open, since the finite intersection of open sets is open.

Now, we want to prove that a subset F' of Y is closed $\iff$ there exists a closed subset F of X such that $F' = Y \cap F$. Suppose $F' \subset Y$ is closed. Then $C_Y(F')$ is open $\iff \exists\, O \subset X$ closed such that $C_Y(F') = Y \cap O \iff C_Y\left(C_Y(F')\right) = C_Y(Y \cap O) \iff F' = C_Y(Y) \cup C_Y(O) = C_Y(O) = Y \cap C_X(O)$.

Lastly, we want to show that $N' \subset Y$ is a neighborhood of $a \in Y \iff$ there exists a neighborhood $N \subset X$ of a such that $N' = Y \cap N$.

($\Rightarrow$) Suppose $N' \subset Y$ is a neighborhood of $a \in Y$. Then since $X \supset Y$, this implies that $X \supset N'$. Therefore, there exists a neighborhood $N \subset X$ such that $N' = Y \cap N$; namely N'.

($\Leftarrow$) Conversely, suppose there exists a neighborhood $N \subset X$ such that $N' = Y \cap N$. Then since Y is a metric space, Y is open, and since $a \in Y$, this implies that there exists a neighborhood $M \subset Y$ of a; hence, Y is a neighborhood of a. Therefore, $N' = Y \cap N$ is a neighborhood of a.

2.7.7

We are told that (Y, d') is a subspace of (X, d); i.e., $Y \subset X$ and $d' = d|_{Y \times Y}$. We want to show that if $\{a_n\}_{n=1}^{\infty} \subset Y$, $a \in Y$, and $\lim\limits_{n\to\infty} a_n = a$ in (Y, d'), then $\lim\limits_{n\to\infty} a_n = a$ in (X, d).

Since $\lim\limits_{n\to\infty} a_n = a$ in (Y, d'), this means that $\forall\, \epsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ such that $d'(a_n, a) < \frac{\epsilon}{2}$ for $n \geq N_1$. Now, suppose there exists $b \in X$ such that $\lim\limits_{n\to\infty} a_n = b$ in (X, d). Then $\forall\, \epsilon > 0$, $\exists\, N_2 \in \mathbb{N}$ such that $d(a_n, a) < \frac{\epsilon}{2}$ for $n \geq N_2$. Now, let $N := \max\{N_1, N_2\}$. Then given $\epsilon > 0$, $n \geq N$ implies that:

$$\begin{aligned} d(a, b) &\leq d(a, a_n) + d(a_n, b) \\ &= d'(a_n, a) + d(a_n, b) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Rightarrow a = b$.

2.7.8

We are told that there exists a sequence of points $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = \sqrt{2}$; that is, given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $n \geq N$ implies that $|a_n - \sqrt{2}| < \frac{\epsilon}{2}$. Hence, if $n, m \geq N$, then $|a_n - a_m| = |a_n - \sqrt{2} + \sqrt{2} - a_m| \leq |a_n - \sqrt{2}| + |\sqrt{2} - a_m| = |a_n - \sqrt{2}| + |a_m - \sqrt{2}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$; thus, $\{a_n\}$ is a Cauchy sequence. Note that $\{a_n\}$ does not converge in $(\mathbb{Q}, d|_{\mathbb{Q}\times\mathbb{Q}})$ since limits of real-valued sequences are unique and $\sqrt{2} \notin \mathbb{Q}$.

2.8

2.8.1

It is straightforward veryifying that H is a vector space over $\mathbb{R}$ with the usual operations of componentwise addition and componentwise scalar multiplication.

Now, let $u, v, w \in H$, and define $A : H \times H \to \mathbb{R}$ as $A(u, v) = \sum_{i=1}^{\infty} u_i v_i$. Then if $\alpha, \beta, \gamma \in H$, we have:

$$A(\alpha u + \beta v, w) = \sum_{i=1}^{\infty}(\alpha u_i + \beta v_i)w_i = \sum_{i=1}^{\infty}(\alpha u_i w_i + \beta v_i w_i) = \alpha \sum_{i=1}^{\infty} u_i w_i + \beta \sum_{i=1}^{\infty} v_i w_i = \alpha A(u, w) + \beta A(v, w)$$

and similarly, we have:

$$A(u, \beta v + \gamma w) = \sum_{i=1}^{\infty} u_i(\beta v_i + \gamma w_i) = \sum_{i=1}^{\infty}(\beta u_i v_i + \gamma u_i w_i) = \beta \sum_{i=1}^{\infty} u_i v_i + \gamma \sum_{i=1}^{\infty} v_i w_i = \beta A(u, v) + \gamma A(u, w)$$

Thus, A is of bilinear form. Moreover, observe that for any $u \in H \setminus \{\mathbf{0}\}$, $A(u, u) = \sum_{i=1}^{\infty} u_i^2 > 0$ since $u_i \geq 0 \; \forall i \in \mathbb{N}$ and there exists atleast one $j \in \mathbb{N}$ such that $u_j \neq 0 \Rightarrow u_j^2 > 0$. Therefore, A is positive definite.

2.8.2

Let $A : V \times V \to \mathbb{R}$ be a positive definite bilinear form on V and define $N : V \to \mathbb{R}$ as $N(v) = [A(v, v)]^{\frac{1}{2}}$; we want to show that N defines a norm on V. Observe that if $v, w \in V$ and $\alpha \in \mathbb{R}$, then we have:

1. $v \neq \mathbf{0} \Rightarrow N(v) = [A(v, v)]^{\frac{1}{2}} = \sqrt{A(v, v)} > 0$ since $A(v, v) > 0$.
2. Note that since A is a bilinear form, $A(\mathbf{0}, \mathbf{0}) = A(0 \cdot \mathbf{0}, \mathbf{0}) = 0 \cdot A(\mathbf{0}, \mathbf{0}) = 0$. Therefore, $v = \mathbf{0} \Rightarrow N(v) = [A(\mathbf{0}, \mathbf{0})]^{\frac{1}{2}} = \sqrt{A(\mathbf{0}, \mathbf{0})} = \sqrt{0} = 0$. Thus, we conclude that $v = \mathbf{0} \iff N(v) = 0$.
3. Observe that

$$\begin{aligned}
[N(v + w)]^2 &= A(v + w, v + w) \\
&= A(v, v) + A(v, w) + A(w, v) + A(w, w) \\
&= N(v)^2 + N(w)^2 + A(v, w) + A(w, v) \\
&\leq N(v)^2 + N(w)^2 + 2A(v, v)A(w, w) \text{ b/c by Schwarz inequality } A(v, w) \leq A(v, v)A(w, w) \\
&= N(v)^2 + N(w)^2 + 2N(v)N(w) \\
&= [N(v) + N(w)]^2 - 2N(v)N(w) + 2N(v)N(w) \\
&= [N(v) + N(w)]^2
\end{aligned}$$

$\Rightarrow N(v + w) \leq N(v) + N(w)$

4. Observe that

$$N(\alpha v) = [A(\alpha v, \alpha v)]^{\frac{1}{2}} = [\alpha A(v, \alpha v)]^{\frac{1}{2}} = [\alpha^2 A(v, v)]^{\frac{1}{2}} = |\alpha| N(v)$$

Therefore, N defines a norm on V.

2.8.3

First we want to show that $d : V \times V \to \mathbb{R}$ defined as $d(u, v) = N(u - v)$ is a metric. Observe that for any $u, v, w \in V$, we have:

1. $d(u, v) = N(u - v) \geq 0$ since N is a norm on the vector space V.
2. $u = v \Rightarrow d(u, v) = d(u, u) = N(u - u) = N(\mathbf{0}) = 0$, and conversely $d(u, v) = N(u - v) = 0 \Rightarrow u - v = \mathbf{0}$. Hence, $d(u, v) = 0 \iff u = v$.
3. $d(u, w) = N(u - w) = N(u - v + v - w) = N\big((u - v) + (v - w)\big) \leq N(u - v) + N(v - w) = d(u, v) + d(v, w)$

$\Rightarrow d$ is a metric.

Now, we want to show that the function $a : V \times V \to V$ defined as $a(u,v) = u + v$ is continuous. Equipping V with the metric above and $V \times V$ with the metric $d' : (V \times V) \times (V \times V) \to \mathbb{R}$ defined as $d'\big((u_1,u_2),(v_1,v_2)\big) = \max\{d(u_1,v_1), d(v_1,v_2)\} = \max\{N(u_1 - v_1), N(u_2 - v_2)\}$, we want to show that $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ s.t. $\forall$ $(u_1,u_2),(v_1,v_2) \in V \times V$ with $d'\big((u_1,u_2),(v_1,v_2)\big) < \delta$, we have $d\big(a(u_1,u_2), a(v_1,v_2)\big) < \epsilon$. Given $\epsilon > 0$, let $\delta := \frac{\epsilon}{2}$. Then observe that for all $(u_1,u_2),(v_1,v_2) \in V \times V$ such that $d'\big((u_1,u_2),(v_1,v_2)\big)$, we have:

$$\begin{aligned} d\big(a(u_1,u_2), a(v_1,v_2)\big) &= d(u_1 + u_2, v_1 + v_2) \\ &= N\big(u_1 + u_2 - (v_1 + v_2)\big) \\ &= N(u_1 - v_1) + N(u_2 - v_2) \\ &\leq 2\max\{N(u_1 - v_1), N(u_2 - v_2)\} \\ &= 2d'\big((u_1,u_2),(v_1,v_2)\big) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

$\Rightarrow a$ is continuous.

Now, we want to show that the function $b : V \to V$ defined as $b(v) = -v$ is continuous; that is, we want to prove that $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $d(u,v) = N(u - v) < \delta$ implies that $d\big(b(u), b(v)\big) = N\big(b(u) - b(v)\big) < \epsilon$. Given $\epsilon > 0$, let $\delta := \epsilon$. Then for all $u, v \in V$ such that $d(u,v) < \delta$, we have:

$$\begin{aligned} d\big((b(u), b(v)\big) &= d(-u, -v) \\ &= N\big(-u - (-v)\big) \\ &= N(v - u) \\ &= N(-1(u - v)) \\ &= |-1|N(u - v) \\ &= d(u,v) \\ &< \delta \\ &= \epsilon \end{aligned}$$

$\Rightarrow b$ is continuous.

Lastly, we want to show that the function $c : \mathbb{R} \times V \to V$ defined as $c(\alpha, v) = \alpha v$ is continuous; that is, we want to prove that $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $d(u,v) = N(u - v) < \delta$ implies that $d\big(c(u), c(v)\big) = N\big(c(u) - c(v)\big) < \epsilon$. Given $\epsilon > 0$ let $\delta := \frac{\epsilon}{|\alpha|}$. Then for all $u, v \in V$ such that $d(u,v) < \delta$, we have:

$$\begin{aligned} d\big((c(u), c(v)\big) &= d(\alpha u, \alpha v) \\ &= N(\alpha u - \alpha v) \\ &= N\big(\alpha(u - v)\big) \\ &= |\alpha|N(u - v) \\ &< \delta \\ &= \epsilon \end{aligned}$$

$\Rightarrow c$ is continuous. And we are done folks!